A Note on the Gutman Index of Jaco Graphs

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Abstract

The concept of the *Gutman index,* denoted $Gut(G)$ was introduced for a connected undirected graph G. In this note we apply the concept to the underlying graphs of the family of Jaco graphs, (*directed graphs by definition*), and decribe a recursive formula for the *Gutman index* $Gut(J^*_{n+1}(x))$. We also determine the *Gutman index* for the trivial *edge-joint* between Jaco graphs.

Keywords: Gutman index, Jaco graph, edge-joint

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1 Introduction

For general reference to notation and concepts of graph theory see [2]. Unless mentioned otherwise, a graph $G = G(V, E)$ on $\nu(G)$ vertices (order of G) with $\epsilon(G)$ edges (size of G) will be a finite undirected and connected simple graph. The degree of a vertex in G is denoted $d_G(v)$ and if the context of G is clear the degree is denoted $d(v)$ for brevity. Also in a directed graph $G^{\rightarrow}$ the degree is $d_{G^{\rightarrow}}(v) = d^+_{G^{\rightarrow}}(v) + d^-_{G^{\rightarrow}}(v)$ or for brevity, $d(v) = d^+(v) + d^-(v)$ if G is clear.

The concept of the Gutman index $Gut(G)$ of a connected undirected graph G was introduced in 1994 by Gutman [4]. It is defined to be $Gut(G) = \sum\limits_{\{v,u\}\subseteq V(G)} d_G(v)d_G(u)d_G(v,u)$, where $d_G(v)$ and $d_G(u)$ are the degree of v and u in G respectively, and $d_G(v,u)$ is the distance between v and u in G. Clearly, if the vertices of G of order n are randomly labeled $v_1, v_2, v_3, ..., v_n$ the definition states that $Gut(G) = \sum\limits_{\ell=1}^{n-1}\sum\limits_{j=\ell+1}^{n} d_G(v_\ell)d_G(v_j)d_G(v_\ell, v_j)$. Worthy results are reported in Andova et al. [1] and Dankelmann et al. [3].

2 The Gutman Index of the Underlying Graph of a Jaco Graph

Despite earlier definitions in respect of the family of Jaco graphs [5, 6], the definitions found in [7] serve as the unifying definitions. For ease of reference some of the important definitions are repeated here.

Definition 2.1. [7] *Let $f(x) = mx + c; x \in \mathbb{N}$, $m, c \in \mathbb{N}_0$. The family of infinite linear Jaco graphs denoted by $\{J_\infty(f(x)) : f(x) = mx + c; x \in \mathbb{N}$ and $m, c \in \mathbb{N}_0\}$ is defined by $V(J_\infty(f(x))) = \{v_i : i \in \mathbb{N}\}$, $A(J_\infty(f(x))) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in A(J_\infty(f(x)))$ if and only if $(f(i) + i) - d^-(v_i) \geq j$.*

Definition 2.2. [7] *The family of finite linear Jaco graphs denoted by $\{J_n(f(x)) : f(x) = mx+c; x \in \mathbb{N}$ and $m, c \in \mathbb{N}_0\}$ is defined by $V(J_n(f(x))) = \{v_i : i \in \mathbb{N}, i \leq n\}$, $A(J_n(f(x))) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j \leq n\}$ and $(v_i, v_j) \in A(J_n(f(x)))$ if and only if $(f(i) + i) - d^-(v_i) \geq j$.*

The reader is referred to [7] for the definition of the *prime Jaconian vertex* and the *Hope graph*. The graph has four fundamental properties which are:
(i) $V(J_\infty(f(x))) = \{v_i : i \in \mathbb{N}\}$ and,
(ii) if v_j is the head of an arc then the tail is always a vertex v_i, $i < j$ and,
(iii) if v_k, for smallest $k \in \mathbb{N}$ is a tail vertex then all vertices v_ℓ, $k < \ell < j$ are tails of arcs to v_j and finally,
(iv) the degree of vertex k is $d(v_k) = f(k)$.

The family of finite directed graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and arcs to vertices) $v_t, t > n$. Hence, trivially $d(v_i) \leq i$ for $i \in \mathbb{N}$. For $m = 0$ and $c \geq 0$ two special classes of disconnected linear Jaco graphs exist. For $c = 0$ the Jaco graph $J_n(0)$ is a null graph (*edgeless graph*) on n vertices. For $c > 0$, the Jaco graph $J_n(c) = \bigcup\limits_{\lfloor \frac{n}{c+1} \rfloor - copies} K^{\rightarrow}_{c+1} \bigcup K^{\rightarrow}_{n-(c+1)\cdot\lfloor \frac{n}{c+1} \rfloor}$. since the Gutman index is defined for connected graphs the bound $m \geq 1$ will apply.

In this note we only consider the case $m = 1$, $c = 0$. The generalisation for $f(x) = mx + c$ in general remains open. Denote the underlying Jaco graph by $J^*_n(f(x))$. A recursive formula of the Gutman index $Gut(J^*_{n+1}(x))$ in terms of $Gut(J^*_n(x))$ is given in the next theorem.

Theorem 2.1. *For the underlying graph $J^*_n(x)$ of a finitie Jaco Graph $J_n(x), n \in \mathbb{N}, n \geq 2$ with Jaconian vertex v_i we have that recursively:*

$$Gut(J^*_{n+1}(x)) = Gut(J^*_n(x)) + \sum_{k=1}^{i}\sum_{t=i+1}^{n} d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_k, v_t) + \sum_{t=i+1}^{n-1}\sum_{q=t+1}^{n}(d_{J^*_n(x)}(v_t) + d_{J^*_n(x)}(v_q)) +$$
$$(n-i)(\sum_{k=1}^{i} d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_k, v_n) + \sum_{t=i+1}^{n} d_{J^*_n(x)}(v_t)) + (n-i-1) + i(n-i).$$

Proof. Consider the underlying Jaco graph, $J^*_n(x), n \in \mathbb{N}, n \geq 2$ with prime Jaconian vertex v_i. Now consider $J^*_{n+1}(x)$. From the definition of a Jaco graph the extension from $J^*_n(x)$ to $J^*_{n+1}(x)$ adds the vertex v_{n+1} and the edges $v_{i+1}v_{n+1}, v_{i+2}v_{n+1}, ..., v_nv_{n+1}$.

Step 1: Consider any ordered pair of vertices $(v_k, v_q)_{k<q}, 1 \leq k \leq i-1$, and $k+1 \leq q \leq i$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J^*_{n+1}(x)}(v_k)d_{J^*_{n+1}(1)}(v_q)d_{J^*_{n+1}(x)}(v_k, v_q) = d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_q)d_{J^*_n(x)}(v_k, v_q).$$

By applying this step $\forall v_k, 1 \leq k \leq i-1$, and $\forall v_q, k+1 \leq q \leq i$ with $k < q$ we obtain:

$$\sum_{k=1}^{i-1}\sum_{q=k+1}^{i} d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_q)d_{J^*_n(x)}(v_k, v_q).$$

Step 2: Consider any vertex $v_k, 1 \leq k \leq i$ and any other vertex $v_t, i+1 \leq t \leq n$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J^*_{n+1}(x)}(v_k)d_{J^*_{n+1}(x)}(v_t)d_{J^*_{n+1}(x)}(v_k, v_t) = d_{J^*_n(x)}(v_k)(d_{J^*_n(x)}(v_t) + 1)d_{J^*_n(x)}(v_k, v_t) =$$

$$d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_t)d_{J^*_n(x)}(v_k, v_t) + d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_k, v_t).$$

By applying this step $\forall v_k, 1 \leq k \leq i$ and $\forall v_t, i+1 \leq t \leq n$, we obtain:

$$\sum_{k=1}^{i}\sum_{t=i+1}^{n} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_t)d_{J_n^*(x)}(v_k,v_t) + \sum_{k=1}^{i}\sum_{t=i+1}^{n} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_k,v_t).$$

Step 3: Consider any two distinct vertices $v_t, v_q, i+1 \leq t \leq n-1$, and $t+1 \leq q \leq n$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J_{n+1}^*(x)}(v_t)d_{J_{n+1}^*(x)}(v_q)d_{J_{n+1}^*(x)}(v_t,v_q) = (d_{J_n^*(x)}(v_t)+1)(d_{J_n^*(x)}(v_q)+1)d_{J_n^*(x)}(v_t,v_q) =$$

$$d_{J_n^*(x)}(v_t)d_{J_n^*(x)}(v_q) + d_{J_n^*(x)}(v_t) + d_{J_n^*(x)}(v_q) + 1.$$

By applying this step $\forall v_t, i+1 \leq t \leq n-1$ and $\forall v_q, t+1 \leq q \leq n$, we obtain:

$$\sum_{t=i+1}^{n-1}\sum_{q=t+1}^{n} d_{J_n^*(x)}(v_t)d_{J_n^*(x)}(v_q) + \sum_{t=i+1}^{n-1}\sum_{q=t+1}^{n} (d_{J_n^*(x)}(v_t) + d_{J_n^*(x)}(v_q)) + (n-i-1).$$

Step 4: Consider any vertex $v_k, 1 \leq k \leq i$ and the vertex v_{n+1}. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J_{n+1}^*(x)}(v_k)d_{J_{n+1}^*(x)}(v_{n+1})d_{J_{n+1}^*(x)}(v_k,v_{n+1}) = d_{J_n^*(x)}(v_k)(n-i)(d_{J_n^*(x)}(v_k,v_n)+1).$$

By applying this step $\forall v_k, 1 \leq k \leq i$ we obtain:

$$\sum_{k=1}^{i} d_{J_n^*(x)}(v_k)(n-i)(d_{J_n^*(x)}(v_k,v_n)+1) = (n-i)\sum_{k=1}^{i} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_k,v_n) + i(n-i).$$

Step 5: Consider any vertex $v_t, i+1 \leq t \leq n$ and the vertex v_{n+1}. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J_{n+1}^*(x)}(v_t)d_{J_{n+1}^*(x)}(v_{n+1})d_{J_{n+1}^*(x)}(v_t,v_{n+1}) = d_{J_n^*(x)}(v_t)(n-i)d_{J_n^*(x)}(v_t,v_n).$$

By applying this step $\forall v_t, i+1 \leq t \leq n$ we obtain:

$$\sum_{t=i+1}^{n} d_{J_n^*(x)}(v_t)(n-i) = (n-i)\sum_{t=i+1}^{n} d_{J_n^*(x)}(v_t).$$

Final summation step: Adding Steps 1 to 5 and noting that:

$$Gut(J_n^*(x)) = \sum_{k=1}^{i-1}\sum_{q=k+1}^{i} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_q)d_{J_n^*(x)}(v_k,v_q) + \sum_{k=1}^{i}\sum_{t=i+1}^{n} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_t)d_{J_n^*(x)}(v_k,v_t) +$$

$\sum_{t=i+1}^{n-1}\sum_{q=t+1}^{n} d_{J_n^*(x)}(v_t)d_{J_n^*(x)}(v_q)$, provides the result:

$$Gut(J^*_{n+1}(x)) = Gut(J^*_n(x)) + \sum_{k=1}^{i}\sum_{t=i+1}^{n} d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_k,v_t) + \sum_{t=i+1}^{n-1}\sum_{q=t+1}^{n}(d_{J^*_n(x)}(v_t)+d_{J^*_n(x)}(v_q)) +$$
$$(n-i)(\sum_{k=1}^{i} d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_k,v_n) + \sum_{t=i+1}^{n} d_{J^*_n(x)}(v_t)) + (n-i-1) + i(n-i). \qquad \square$$

3 The Gutman Index of the Edge-joint between $J^*_n(x), n \in \mathbb{N}$ and $J^*_m(x), m \in \mathbb{N}$

The concept of an *edge-joint* between two simple undirected graphs G and H is defined below.

Definition 3.1. *The edge-joint of two simple undirected graphs G and H is the graph obtained by linking the edge vu, $v \in V(G), u \in V(H)$ and denoted, $G \rightsquigarrow_{vu} H$.*

Note: $G \rightsquigarrow_{vu} H = G \cup H + vu$, $v \in V(G), u \in V(H)$.

The next theorem provides $Gut(J^*_n(x) \rightsquigarrow_{v_1u_1} J^*_m(x))$ in terms of $Gut(J^*_n(x))$ and $Gut(J^*_m(x))$. The edge-joint $J^*_n(x) \rightsquigarrow_{v_1u_1} J^*_m(x)$ is called *trivial.* Edge-joints $J^*_n(x) \rightsquigarrow_{v_iu_j} J^*_m(x)$, $i \neq 1$ or $j \neq 1$ are called *non-trivial.* For families (classes) of graphs such as paths P_n, cycles C_n, complete graphs K_n, Jaco graphs $J_n(f(x))$, etc, the notation is abbreviated as $P_n \rightsquigarrow_{vu} P_m = P^{\rightsquigarrow uv}_{n,m}$ and $J^*_n(f(x)) \rightsquigarrow_{v_iu_j} J^*_m(f(x)) = J^{\rightsquigarrow v_iu_j}_{n,m}$, etc.

Theorem 3.1. *For the underlying graphs $J^*_n(x)$ and $J^*_m(x)$ of the finitie Jaco Graphs $J_n(x), J_m(x)$, $n, m \in \mathbb{N}$ and $n \geq m \geq 2$:*

$$Gut(J^*_n(x) \rightsquigarrow_{v_1u_1} J^*_m(x)) = Gut(J^{\rightsquigarrow v_1u_1}_{n,m}) = Gut(J^*_n(x)) + Gut(J^*_m(x)) + \sum_{\ell=2}^{n} d_{J^*_n(x)}(v_\ell)d_{J^*_n(x)}(v_1,v_\ell) +$$
$$\sum_{s=2}^{m} d_{J^*_m(x)}(u_s)d_{J^*_m(x)}(u_1,u_s) + \sum_{t=2}^{m}(d_{J^*_n(x)}(v_1)+1)d_{J^*_m(x)}(u_t)(d_{J^*_m(x)}(u_1,u_t)+1) +$$
$$\sum_{k=2}^{n}\sum_{t=2}^{m} d_{J^*_n(x)}(v_k)d_{J^*_m(x)}(u_t)(d_{J^*_n(x)}(v_1,v_k) + d_{J^*_m(x)}(u_1,u_t)+1) + 4.$$

Proof. Consider the underlying Jaco graphs, $J^*_n(x), J^*_m(x)$, with $n, m \in \mathbb{N}$ and $n \geq m \geq 2$ with $J_m(x)$ having prime Jaconian vertex u_i. Also label the vertices of $J^*_n(x)$ and $J^*_m(x)$; $v_1, v_2, v_3, ..., v_n$ and $u_1, u_2, u_3, ..., u_m$, respectively. Consider $J^{\rightsquigarrow v_1u_1}_{n,m} = J^*_n(x) \cup J^*_m(x) + v_1u_1$. Without loss of generality apply the piecewise definition:

$$Gut(J_{n,m}^{\rightsquigarrow v_1u_1}) = \sum_{k=1}^{n-1}\sum_{\ell=k+1}^{n} d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_\ell)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k,v_\ell)+\sum_{t=1}^{m-1}\sum_{s=t+1}^{m} d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_s)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t,u_s)+$$

$$\sum_{k=1}^{n}\sum_{t=2}^{m} d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k,u_t) + d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_1)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1,u_1).$$

Step 1(a): Consider vertex v_1 and vertex $v_\ell, 2 \leq \ell \leq n$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_\ell)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1,v_\ell) = (d_{J_n^*(x)}(v_1)+1)d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell) =$$

$$d_{J_n^*(x)}(v_1)d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell) + d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell).$$

By applying this step $\forall v_\ell, 2 \leq \ell \leq n$ we obtain:

$$\sum_{\ell=2}^{n} d_{J_n^*(x)}(v_1)d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell) + \sum_{\ell=2}^{n} d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell).$$

Step 1(b): For all ordered pairs of vertices $(v_k, v_\ell)_{k<\ell}$ with $2 \leq k \leq n-1$ and $3 \leq \ell \leq n$ we have that:

$$\sum_{k=2}^{n-1}\sum_{\ell=k+1}^{n} d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_\ell)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k,v_\ell) = \sum_{k=2}^{n-1}\sum_{\ell=k+1}^{n} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_k,v_\ell).$$

By applying this step $\forall (v_k, v_\ell)_{k<\ell}$, $1 \leq k \leq n-1$ and $2 \leq \ell \leq n$, we obtain:

$$\sum_{k=1}^{n-1}\sum_{\ell=k+1}^{n} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_k,v_\ell) + \sum_{\ell=2}^{n} d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell) =$$
$$Gut(J_n^*(x)) + \sum_{\ell=2}^{n} d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell).$$

Step 2: Similar to Step 1 we have that:

$$\sum_{t=1}^{m-1}\sum_{s=t+1}^{m} d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_s)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t,u_s) = \sum_{t=1}^{m-1}\sum_{s=t+1}^{m} d_{J_m^*(x)}(u_t)d_{J_m^*(x)}(u_s)d_{J_m^*(x)}(u_t,u_s) +$$
$$\sum_{s=2}^{m} d_{J_m^*(x)}(u_s)d_{J_m^*(x)}(u_1,u_s) = Gut(J_m^*(x)) + \sum_{s=2}^{m} d_{J_m^*(x)}(u_s)d_{J_m^*(x)}(u_1,u_s).$$

Step 3: To conclude this step we will provide the next partial summation as a piecewise summation, to be:

$$\sum_{k=1}^{n}\sum_{t=2}^{m} d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k,u_t) = \sum_{t=2}^{m} d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1,u_t) +$$

$\sum_{k=2}^{n}\sum_{t=2}^{m} d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k,u_t).$

Step 3(a): Consider vertex v_1 and vertex $u_t, 2 \leq t \leq m$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1,u_t) = (d_{J_n^*(x)}(v_1)+1)d_{J_m^*(x)}(u_t)(d_{J_m^*(x)}(u_1,u_t)+1).$

By applying this step $\forall u_t, 2 \leq t \leq m$ we obtain:

$\sum_{t=2}^{m}(d_{J_n^*(x)}(v_1)+1)d_{J_m^*(x)}(u_t)(d_{J_m^*(x)}(u_1,u_t)+1).$

Step 3(b): Consider vertex $v_k, 2 \leq k \leq n$ and vertex $u_t, 2 \leq t \leq m$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k,u_t) = d_{J_n^*(x)}(v_k)d_{J_m^*(x)}(u_t)(d_{J_n^*(x)}(v_1,v_k)+d_{J_m^*(x)}(u_1,u_t)+1).$

By applying the step $\forall v_k, 2 \leq k \leq n$ and $\forall u_t, 2 \leq t \leq m$, we obtain:

$\sum_{k+2}^{n}\sum_{t=2}^{m} d_{J_n^*(x)}(v_k)d_{J_m^*(x)}(u_t)(d_{J_n^*(x)}(v_1,v_k)+d_{J_m^*(x)}(u_1,u_t)+1).$

Step 4: It is easy to see that:

$d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_1)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1,u_1) = 4.$

Final summation step: Adding Steps 1 to 4 provides the result:

$Gut(J_{n,m}^{\rightsquigarrow v_1u_1}) = Gut(J_n^*(x))+Gut(J_m^*(x))+\sum_{\ell=2}^{n} d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell)+\sum_{s=2}^{m} d_{J_m^*(x)}(u_s)d_{J_m^*(x)}(u_1,u_s)+ \sum_{t=2}^{m}(d_{J_n^*(x)}(v_1)+1)d_{J_m^*(x)}(u_t)(d_{J_m^*(x)}(u_1,u_t)+1) + \sum_{k=2}^{n}\sum_{t=2}^{m} d_{J_n^*(x)}(v_k)d_{J_m^*(x)}(u_t)(d_{J_n^*(x)}(v_1,v_k) + d_{J_m^*(x)}(u_1,u_t)+1)+4.$ □

4 Conclusion

For the simple case $f(x) = x$ the calculation of the Gutman index for Jaco graph and the edge-joint between them is immediately complicated. Finding a result similar to Theorem 3.1 for $J_n^*(x) \rightsquigarrow_{v_i u_j} J_m^*(x)$, $i \neq 1$ or $j \neq 1$ *(non-trivial edge-joints)* remains open. The single most important challege is to find a closed formula for the number of edges in $J_n(x)$. Such closed formula will enable finding a closed formula for distances between given vertices and a simplied formula for many invariants of Jaco graphs might result from such finding. Hence, important open questions remain such as: *Is there a closed formula for the number of edges of* $J_n(x)$, $n \in \mathbb{N}$? *Is there a closed formula for the cardinality of the Jaconian set* $\mathbb{J}(J_n(x))$ *of* $J_n(x), n \in \mathbb{N}$? *Is there a closed formula for* $d_{J_n^*(x)}(v_1, v_n)$ *in* $J_n^*(x), n \in \mathbb{N}$?. Refer to [7] for further reading.

References (Limited)

[1] V. Andova, D. Dimitrov, J. Fink, R. Škrekovski, *Bounds on Gutman Index*, MATCH Communications in Mathematical and in Computer Chemistry, Vol 67 (2012), pp 515-524.
[2] J.A. Bondy, U.S.R. Murty,*Graph Theory with Applications,* Macmillan Press, London, (1976).
[3] P. Dankelmann, I. Gutman, S. Mukwembi, H.C. Swart, *The edge-Wiener index of a graph*, Discrete Mathematics, Vol 309 (2009), pp 3452-3457.
[4] I. Gutman, *Selected properties of the Schultz molecular topological index*, Journal of Chemical Information and Computer Sciences, Vol 34, (1994) pp 1087-1089.
[5] J. Kok, P. Fisher, B. Wilkens, M. Mabula, V. Mukungunugwa, *Characteristics of Finite Jaco Graphs,* $J_n(1), n \in \mathbb{N}$, arXiv: 1404.0484v1 [math.CO], 2 April 2014.
[6] J. Kok, P. Fisher, B. Wilkens, M. Mabula, V. Mukungunugwa, *Characteristics of Jaco Graphs,* $J_\infty(a), a \in \mathbb{N}$, arXiv: 1404.1714v1 [math.CO], 7 April 2014.
[7] J. Kok, C. Susanth, S.J. Kalayathankal, *A Study on Linear Jaco Graphs*, arXiv:1506.06538v1, [math.CO], 22 June 2015, Journal of Informatics and Mathematical Sciences (Accepted).